CHERIAN'S BIVARIATE GAMMA DISTRIBUTION AS A MODEL FOR DROUGHT DATA

LA DISTRIBUCIÓN GAMA BIVARIADA DE CHERIAN COMO UN MODELO PARA DATOS DE SEQUÍA

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ABSTRACT

Models for drought are of primary importance for the agricultural sciences. In this note, a drought application is described by deriving the exact distribution of $W=X/(X+Y)$ and the corresponding moment properties when X and Y follow Cherian's bivariate gamma distribution. Drought data from the State of Nebraska are used.

Key words: Cherian's bivariate gamma distribution, drought modeling, ratio of random variables.

INTRODUCTION

Having flexible models for drought is of primary importance for the agricultural sciences. The aim of this note is to provide a drought application by deriving the exact distribution of $W = X/(X+Y)$ when (X, Y) follows Cherian (1941)'s bivariate gamma distribution given by the joint probability density function (pdf)

$$f(x,y) = \frac{\exp(-x-y)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)}$$

$$\int_0^{\min(x,y)} (x-z)^{\theta_1-1} (y-z)^{\theta_2-1} z^{\theta_3-1} \exp(z)\, dz \qquad (1)$$

for $x>0$, $y>0$, $\theta_1>0$, $\theta_2>0$, and $\theta_3>0$.

We have chosen this model because bivariate gamma distributions are some of the most popular models for hydrological processes Phatarfod (1976), Prekopa and Szantai (1978), Izawa (1965), Smith and Adelfang (1981), Yue (2001), Yue *et al.* (2001), Shiau *et al*. (2006). Ratios of random variables arise naturally with respect to drought modeling: if X and Y denote the drought duration and the successive non

RESUMEN

Los modelos para sequía son de importancia fundamental para las ciencias agrícolas. En esta nota se describe una aplicación a la sequía derivando la distribución exacta de $W=X/(X+Y)$ y las propiedades de momentos correspondientes cuando X y Y siguen la distribución gama bivariada de Cherian. Se emplean los datos de sequía del Estado de Nebraska.

Palabras clave: Distribución gama bivariada de Cherian, modelaje de sequía, razón de variables aleatorias.

INTRODUCCIÓN

Disponer de modelos flexibles para la sequía es de vital importancia para las ciencias agrícolas. El propósito de esta nota es proporcionar una aplicación a datos de sequía mediante la derivación de la distribución exacta de $W = X/(X+Y)$, cuando (X, Y) sigue la distribución gama bivariada de Cherian (1941) dada por la función de densidad de probabilidad conjunta (fdp)

$$f(x,y) = \frac{\exp(-x-y)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)}$$

$$\int_0^{\min(x,y)} (x-z)^{\theta_1-1} (y-z)^{\theta_2-1} z^{\theta_3-1} \exp(z)\, dz \qquad (1)$$

para $x>0$, $y>0$, $\theta_1>0$, $\theta_2>0$, y $\theta_3>0$.

Hemos escogido este modelo porque las distribuciones gama bivariadas están entre los modelos más populares para los procesos hidrológicos (Phataford (1976), Prekopa y Szantai (1978), Izawa (1965), Smith y Adelfang (1981), Yue (2001), Yue *et al.* (2001), Shiau *et al.* (2006). Cuando se modelan datos de sequía aparecen de manera natural razones de variables aleatorias: si X y Y denotan la duración de la sequía y la sucesiva duración de no sequía, entonces $W = X/(X+Y)$ representarán la proporción de l eventos de sequía (ver Sección 4).

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drought duration then $W = X/(X+Y)$ will represent the proportion of drought events (see Section 4).

This note is organized as follows. First, explicit expressions are derived for the pdf and moments of $W = X/(X+Y)$. Next, an application of the results to drought data from the State of Nebraska is provided. The calculations of this note involve the incomplete beta function defined by

$$B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\,dt$$

and the Appell function of the first kind defined by

$$F_1(a,b,c,d;x,y) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(a)_{k+l}(b)_k(c)_l}{(d)_{k+l}} \frac{x^k y^l}{k!\,l!},$$

where $(e)_k = e(e+1)\cdots(e+k-1)$ denotes the ascending factorial. It is also required the following lemma.

Lemma 1

Prudnikov *et al.*, 1986, volume 1 (Equation 2.2.8.5). For $a>0$, $\alpha>0$, $\beta>0$, $a|u|<1$ and $a|v|<1$,

$$\int_0^a x^{\alpha-1}(a-x)^{\beta-1}(1-ux)^{-\rho}(1-vx)^{-\lambda}\,dx$$

$$= a^{\alpha+\beta-1}B(\alpha,\beta)F_1(\alpha,\rho,\lambda,\alpha+\beta;ua,va)$$

The properties of the above special functions can be found in Prudnikov *et al.* (1986) and Gradshteyn and Ryzhik (2000).

Probability density function

Theorem 1 derives the pdf of $W = X/(X+Y)$ when *X* and *Y* are distributed according to (1).

Theorem 1

If *X* and *Y* are jointly distributed according to (1) then

$$f_W(w) = \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_2)\Gamma(\theta_1+\theta_3)} w^{\theta_1+\theta_3-1}(1-w)^{\theta_2-1}$$
$$\times F_1\left(\theta_1, 1-\theta_2, \theta_1+\theta_2+\theta_3, \theta_1+\theta_3; \frac{w}{1-w}, w\right) \quad (2)$$

Esta nota está organizada como sigue. Primero se derivan expresiones explícitas para la fdp y los momentos de $W = X/(X+Y)$. Posteriormente se presenta una aplicación de los resultados a datos de sequía del Estado de Nebraska. Los cálculos de esta nota incluyen la función beta incompleta definida por

$$B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\,dt$$

y la función Appell del primer tipo definida por

$$F_1(a,b,c,d;x,y) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(a)_{k+l}(b)_k(c)_l}{(d)_{k+l}} \frac{x^k y^l}{k!\,l!},$$

donde $(e)_k = e(e+1)\cdots(e+k-1)$ indica el factorial ascendente. Se requiere también el siguiente lema.

Lema 1

Prudnikov *et al.*, 1986, volumen 1, ecuación 2.2.8.5. For $a>0$, $\alpha>0$, $\beta>0$, $a|u|<1$ and $a|v|<1$,

$$\int_0^a x^{\alpha-1}(a-x)^{\beta-1}(1-ux)^{-\rho}(1-vx)^{-\lambda}\,dx$$

$$= a^{\alpha+\beta-1}B(\alpha,\beta)F_1(\alpha,\rho,\lambda,\alpha+\beta;ua,va)$$

Las propiedades de las funciones especiales previamente citadas pueden encontrarse en Prudnikov *et al.* (1986) y Gradshteyn y Ryzhik (2000).

Función de densidad de probabilidades

En el teorema 1 se deriva la fdp de $W = X/(X+Y)$ cuando *X* y *Y* se distribuyen de acuerdo con (1).

Teorema 1

Si *X* y *Y* tienen la distribución conjunta establecida en (1) entonces

$$f_W(w) = \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_2)\Gamma(\theta_1+\theta_3)} w^{\theta_1+\theta_3-1}(1-w)^{\theta_2-1}$$
$$\times F_1\left(\theta_1, 1-\theta_2, \theta_1+\theta_2+\theta_3, \theta_1+\theta_3; \frac{w}{1-w}, w\right) \quad (2)$$

si $0<w\leq ½$, y

if $0 < w \leq ½$, and

$$f_W(w) = \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_2)\Gamma(\theta_1+\theta_3)} w^{\theta_1-1}(1-w)^{\theta_2+\theta_3-1} \times F_1\left(\theta_3, 1-\theta_1, \theta_1+\theta_2+\theta_3, \theta_2+\theta_3; \frac{1-w}{w}, 1-w\right) \qquad (3)$$

if $½ < w \leq 1$.

Proof

From (1), the joint pdf of $(R, W) = (X+Y, X/R)$ becomes

$$f(r,w) = \frac{r\exp(-r)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} \int_0^{r\min(w,1-w)} (rw-z)^{\theta_1-1}(r(1-w)-z)^{\theta_2-1} z^{\theta_3-1}\exp(z)\,dz. \qquad (4)$$

After setting $v=z/r$, (4) can be written as

$$f(r,w) = \frac{r^{\theta_1+\theta_2+\theta_3-1}\exp(-r)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} \int_0^{\min(w,1-w)} (w-v)^{\theta_1-1}(1-w-v)^{\theta_2-1} v^{\theta_3-1}\exp(rv)\,dv.$$

Consider the cases $w \leq ½$ and $w > ½$ separately. If $w \leq ½$ then the pdf of W can be written as

$$f_W(w) = \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} \int_0^{\min(w,1-w)} \frac{(w-v)^{\theta_1-1}(1-w-v)^{\theta_2-1} v^{\theta_3-1}}{(1-v)^{\theta_1+\theta_2+\theta_3}}\,dv$$

$$= \frac{\Gamma(\theta_1+\theta_2+\theta_3) w^{\theta_1+\theta_3-1}(1-w)^{\theta_2-1}}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} \times \int_0^1 (1-u)^{\theta_1-1}\left(1-\frac{wu}{1-w}\right)^{\theta_2-1} u^{\theta_3-1}(1-wu)^{-(\theta_1+\theta_2+\theta_3)}\,du$$

$$= \frac{\Gamma(\theta_1+\theta_2+\theta_3) w^{\theta_1+\theta_3-1}(1-w)^{\theta_2-1}}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} \times B(\theta_1,\theta_3) F_1\left(\theta_1, 1-\theta_2, \theta_1+\theta_2+\theta_3, \theta_1+\theta_3; \frac{w}{1-w}, w\right), \qquad (5)$$

$$f_W(w) = \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_2)\Gamma(\theta_1+\theta_3)} w^{\theta_1-1}(1-w)^{\theta_2+\theta_3-1} \times F_1\left(\theta_3, 1-\theta_1, \theta_1+\theta_2+\theta_3, \theta_2+\theta_3; \frac{1-w}{w}, 1-w\right) \qquad (3)$$

si $½ < w \leq 1$.

Demostración

De (1), la fdp conjunta de $(R, W) = (X+Y, X/R)$ es

$$f(r,w) = \frac{r\exp(-r)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} \int_0^{r\min(w,1-w)} (rw-z)^{\theta_1-1}(r(1-w)-z)^{\theta_2-1} z^{\theta_3-1}\exp(z)\,dz. \qquad (4)$$

Definiendo $v=z/r$, (4) puede escribirse como

$$f(r,w) = \frac{r^{\theta_1+\theta_2+\theta_3-1}\exp(-r)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} \int_0^{\min(w,1-w)} (w-v)^{\theta_1-1}(1-w-v)^{\theta_2-1} v^{\theta_3-1}\exp(rv)\,dv.$$

Considérense separadamente los casos $w \leq ½$ y $w > ½$. Si $w \leq ½$ entonces la fdp de W puede escribirse como

$$f_W(w) = \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} \int_0^{\min(w,1-w)} \frac{(w-v)^{\theta_1-1}(1-w-v)^{\theta_2-1} v^{\theta_3-1}}{(1-v)^{\theta_1+\theta_2+\theta_3}}\,dv$$

$$= \frac{\Gamma(\theta_1+\theta_2+\theta_3) w^{\theta_1+\theta_3-1}(1-w)^{\theta_2-1}}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} \times \int_0^1 (1-u)^{\theta_1-1}\left(1-\frac{wu}{1-w}\right)^{\theta_2-1} u^{\theta_3-1}(1-wu)^{-(\theta_1+\theta_2+\theta_3)}\,du$$

$$= \frac{\Gamma(\theta_1+\theta_2+\theta_3) w^{\theta_1+\theta_3-1}(1-w)^{\theta_2-1}}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} \times B(\theta_1,\theta_3) F_1\left(\theta_1, 1-\theta_2, \theta_1+\theta_2+\theta_3, \theta_1+\theta_3; \frac{w}{1-w}, w\right), \qquad (5)$$

which follows by setting $u=v/w$ and appliying Lemma 1. This establishes the result in (2). The result in (3) can be established similary.

Using special properties of the Appell function of the first kind, one can derive elementary forms for the pdfs in (2) and (3). This is illustrated in the corollary below.

Corollary 1

If X and Y are jointly distributed according to (1) and if $\theta_1 \geq 1$, $\theta_2 \geq 1$ and $\theta_3 \geq 1$ are integers then

$$f_{W(w)} = \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} \sum_{i=0}^{\theta_1-1}\sum_{j=0}^{\theta_2-1}\sum_{k=0}^{\theta_3-1} \binom{\theta_1-1}{i}\binom{\theta_2-1}{j}$$

$$\binom{\theta_3-1}{k}(-1)^{i+j+k+\theta_1+\theta_2-2} \times (1-w)^{\theta_1-1-i}\, w^{\theta_2-1-j}\delta_1$$

$$\left(i+j+k-\theta_1-\theta_2-\theta_3\right)$$

if $0 < w \leq ½$, and

$$f_{W(w)} = \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} \sum_{i=0}^{\theta_1-1}\sum_{j=0}^{\theta_2-1}\sum_{k=0}^{\theta_3-1} \binom{\theta_1-1}{i}\binom{\theta_2-1}{j}$$

$$\binom{\theta_3-1}{k}(-1)^{i+j+k+\theta_1+\theta_2-2} \times (1-w)^{\theta_1-1-i}\, w^{\theta_2-1-j}\delta_2$$

$$\left(i+j+k-\theta_1-\theta_2-\theta_3\right)$$

if $½ < w \leq 1$, where $\delta_1(m) = \left\{1-(1-w)^{m+1}\right\}$ $/(m+1)$ if $m \neq -1, \delta_2(m) = \left\{1-w^{m+1}\right\}/(m+1)$ if $m \neq -1, \delta_1(m) = -\log(1-w)$ and $\delta_2(m) = -\log w$.

MOMENTS

Here, we derive the moments of $W = X/(X+Y)$ when *X* and *Y* are distributed according to (1).

Theorem 2

If *X* and *Y* are jointly distributed according to (1) then

que se sigue de definir $u=v/w$ y aplicar el Lema 1. Esto establece el resultado en (2). El resultado en (3) puede establecerse de manera similar.

Empleando propiedades especiales de la función Appell del primer tipo, pueden derivarse formas elementales para las fpd en (2) y (3). Esto se ilustra en el siguiente corolario.

Corolario 1

Si *X* y *Y* se distribuyen conjuntamente de acuerdo con (1) y si $\theta_1 \geq 1$, $\theta_2 \geq 1$ y $\theta_3 \geq 1$ son enteros, entonces

$$f_{W(w)} = \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} \sum_{i=0}^{\theta_1-1}\sum_{j=0}^{\theta_2-1}\sum_{k=0}^{\theta_3-1} \binom{\theta_1-1}{i}\binom{\theta_2-1}{j}$$

$$\binom{\theta_3-1}{k}(-1)^{i+j+k+\theta_1+\theta_2-2} \times (1-w)^{\theta_1-1-i}\, w^{\theta_2-1-j}\delta_1$$

$$\left(i+j+k-\theta_1-\theta_2-\theta_3\right)$$

si $0 < w \leq ½$, y

$$f_{W(w)} = \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} \sum_{i=0}^{\theta_1-1}\sum_{j=0}^{\theta_2-1}\sum_{k=0}^{\theta_3-1} \binom{\theta_1-1}{i}\binom{\theta_2-1}{j}$$

$$\binom{\theta_3-1}{k}(-1)^{i+j+k+\theta_1+\theta_2-2} \times (1-w)^{\theta_1-1-i}\, w^{\theta_2-1-j}\delta_2$$

$$\left(i+j+k-\theta_1-\theta_2-\theta_3\right)$$

si $½ < w \leq 1$, donde $\delta_1(m) = \left\{1-(1-w)^{m+1}\right\}$ $/(m+1)$ if $m \neq -1, \delta_2(m) = \left\{1-w^{m+1}\right\}/(m+1)$ if $m \neq -1, \delta_1(m) = -\log(1-w)$ y $\delta_2(m) = -\log w$.

MOMENTOS

Aquí se derivan los momentos de $W = X/(X+Y)$ cuando X y Y se distribuyen de acuerdo con (1).

Teorema 2

Si *X* y *Y* se distribuyen conjuntamente de acuerdo con (1) entonces

$$E\left(W^n\right) = \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(\theta_1)_{k+l}(1-\theta_2)_k(\theta_1+\theta_2+\theta_3)_l}{(\theta_1+\theta_3)_{k+l}} \times \frac{B_{1/2}(n+\theta_1+\theta_3+k+l,\theta_2-k)}{k!l!} + \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(\theta_3)_{k+l}(1-\theta_1)_k(\theta_1+\theta_2+\theta_3)_l}{(\theta_2+\theta_3)_{k+l}} \times \frac{1-B_{1/2}(n+\theta_1-k,\theta_2+\theta_3+k+l)}{k!l!} \tag{6}$$

for n≥1.

Proof

$$E\left(W^n\right) = \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(\theta_1)_{k+l}(1-\theta_2)_k(\theta_1+\theta_2+\theta_3)_l I_1(k,l)}{(\theta_1+\theta_3)_{k+l}\, k!l!} + \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_1)\Gamma(\theta_2+\theta_3)} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(\theta_3)_{k+l}(1-\theta_1)_k(\theta_1+\theta_2+\theta_3)_l I_2(k,l)}{(\theta_2+\theta_3)_{k+l}\, k!l!} \tag{7}$$

where

$$I_1(k,l) = \int_0^{1/2} w^{k+l+\theta_1+\theta_3-1}(1-w)^{\theta_2-k-1}\,dw$$

and

$$I_2(k,l) = \int_{1/2}^{1} w^{\theta_1-k-1}(1-w)^{k+l+\theta_2+\theta_3-1}\,dw$$

Using the definition of the incomplete beta function, it follows that

$$E\left(W^n\right) = \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(\theta_1)_{k+l}(1-\theta_2)_k(\theta_1+\theta_2+\theta_3)_l}{(\theta_1+\theta_3)_{k+l}} \times \frac{B_{1/2}(n+\theta_1+\theta_3+k+l,\theta_2-k)}{k!l!} + \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(\theta_3)_{k+l}(1-\theta_1)_k(\theta_1+\theta_2+\theta_3)_l}{(\theta_2+\theta_3)_{k+l}} \times \frac{1-B_{1/2}(n+\theta_1-k,\theta_2+\theta_3+k+l)}{k!l!} \tag{6}$$

para n≥1.

Demostración

$$E\left(W^n\right) = \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(\theta_1)_{k+l}(1-\theta_2)_k(\theta_1+\theta_2+\theta_3)_l I_1(k,l)}{(\theta_1+\theta_3)_{k+l}\, k!l!} + \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_1)\Gamma(\theta_2+\theta_3)} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(\theta_3)_{k+l}(1-\theta_1)_k(\theta_1+\theta_2+\theta_3)_l I_2(k,l)}{(\theta_2+\theta_3)_{k+l}\, k!l!} \tag{7}$$

donde

$$I_1(k,l) = \int_0^{1/2} w^{k+l+\theta_1+\theta_3-1}(1-w)^{\theta_2-k-1}\,dw$$

y

$$I_2(k,l) = \int_{1/2}^{1} w^{\theta_1-k-1}(1-w)^{k+l+\theta_2+\theta_3-1}\,dw$$

Utilizando la definición de la función beta incompleta, tenemos que

$$I_1(k,l) = B_{\frac{1}{2}}(n+\theta_1+\theta_3+k+l,\theta_2-k) \tag{8}$$

and

$$I_2(k,l) = 1 - B_{\frac{1}{2}}(n+\theta_1-k,\theta_2+\theta_3+k+l) \tag{9}$$

The result of the theorem follows by (7), (8) and (9).

APPLICATION

Here, we return to the drought problem discussed previously and provide an application of the model given by (1). We use the drought data from the State of Nebraska. The data are taken from the monthly modified Palmer Drought Severity Index (PDSI) from January 1895 to December 2004. A drought is said to have happened when PDSI is below zero. Defined by the theory of runs (Yevjevich, 1967), some statistics of the observed drought for the eight climatic divisions of Nebraska are summarized in Table 1. The State of Nebraska has eight climatic divisions numbered 1, 2, 3, 5, 6, 7, 8 and 9 –there is no climate division 4 for Nebraska.

Using the PDSI, data on drought duration and non-drought duration were obtained for each climate division. The interest is in determining the distribution of the proportion of droughts: *W*=drought duration/(drought duration+non-drought duration). This distribution was determined by fitting the model given by (1) to the observed values of drought duration (*X*), non-drought duration (*Y*) and using equations (2) and (3) to compute the fitted pdf. The fitting of (1) was performed by the method of maximum likelihood. The quasi-Newton algorithm nlm in the R software package (Dennis and Schnabel, 1983; Schnabel *et al*, 1985; Ihaka and Gentleman, 1996) was used to maximize the likelihood. Since (1) has gamma marginals with unit scale parameters, the data on *X* and *Y* were transformed to $\left(\bar{x}X/s_x^2, \bar{y}Y/s_y^2\right)$, prior to the fitting, where $(\bar{x},\bar{y})$ and $\left(s_x^2,s_y^2\right)$ are the sample means and variances.

The fitted pdf of *W* for the eight climate divisions are shown in Figure 1. The estimated values of the moments *E*(*W*), along with their 95% percentile-based confidence intervals are given in Table 2. These estimates were computed using equation (6). It is evident from both the figure and the table that there is little difference between the climate divisions. This is what one would expect given the geography of the State of Nebraska.

$$I_1(k,l) = B_{\frac{1}{2}}(n+\theta_1+\theta_3+k+l,\theta_2-k) \tag{8}$$

y

$$I_2(k,l) = 1 - B_{\frac{1}{2}}(n+\theta_1-k,\theta_2+\theta_3+k+l) \tag{9}$$

El resultado del teorema se sigue de (7), (8) y (9).

APLICACIÓN

Retomando el problema de sequía discutido previamente, se presenta una aplicación del modelo dado por (1). Se usan datos de sequía del Estado de Nebraska. Los datos se toman del Índice Modificado de Severidad de Sequía de Palmer (IMSSP) de enero de 1895 a Diciembre de 2004. Se considera que ha habido una sequía cuando el IMSSP está por debajo de cero. Definidas por la teoría de las rachas (Yevjevich, 1967), algunas de las estadísticas de la sequías observadas para las ocho divisiones climáticas de Nebraska se resumen en el Cuadro 1. El estado de Nebraska tiene ocho divisiones climáticas enumeradas como 1, 2, 3, 5, 6, 7, 8 y 9 –no hay división climática 4 para Nebraska.

Con base en el IMSSP, se obtuvieron datos sobre la duración de los periodos con y sin sequía para cada división climática. El interés se centra en la determinación de la distribución de la proporción de las sequías: W=duración de la sequía/(duración de la sequía+duración de la no-sequía). Esta distribución se determinó aplicando el modelo en (1) a los valores observados de duración de sequía (*X*), duración de no-sequía (*Y*) y empleando las ecuaciones (2) y (3) para calcular la fdp ajustada. El ajuste de (1) se hizo con el método de máxima verosimilitud. Se usó el algoritmo quasi-Newton del paquete de software R (Dennis y Schnabel, 1983; Schnabel *et al.*, 1985; Ihaka y Gentleman, 1996) para maximizar la verosimilitud. Puesto que (1) tiene marginales gama con parámetros de escala unitarios, los datos para *X* y *Y* se transformaron a $\left(\bar{x}X/s_x^2, \bar{y}Y/s_y^2\right)$, antes del ajuste, donde $(\bar{x},\bar{y})$ y $\left(s_x^2,s_y^2\right)$ son las medias y varianzas muestrales.

En la Figura 1 se muestran las fdp de W ajustadas para las ocho divisiones climáticas. Los valores estimados de los momentos *E*(*W*), junto con sus intervalos de confianza al 95% se muestran en el Cuadro 2. Estas estimaciones se calcularon utilizando la ecuación (6). Partiendo tanto de la figura como del cuadro, es evidente que hay poca diferencia entre las divisiones

Table 1. Basic drought statistics for Nebraska PDSI data.
Cuadro 1. Estadísticas básicas de sequía para los datos del IMSSP (Nebraska).

Climate division	Number of droughts	Drought frequency (number/year)	Mean drought duration (months)
1	83	0.75	6.0
2	66	0.60	8.6
3	89	0.81	6.3
5	81	0.74	6.3
6	90	0.82	6.3
7	81	0.74	6.1
8	76	0.69	6.5
9	74	0.67	7.5

Table 2. Expected values of the ratio.
Cuadro 2. Valores esperados de las proporciones.

Climate division	E(W) (95% CI)
1	0.492 (0.011, 0.977)
2	0.494 (0.014, 0.978)
3	0.487 (0.015, 0.976)
5	0.454 (0.011, 0.978)
6	0.473 (0.017, 0.971)
7	0.463 (0.009, 0.979)
8	0.417 (0.010, 0.977)
9	0.507 (0.018, 0.968)

climáticas. Esto es lo que podría esperarse dada la geografía del Estado de Nebraska.

—Fin de la versión en Español—

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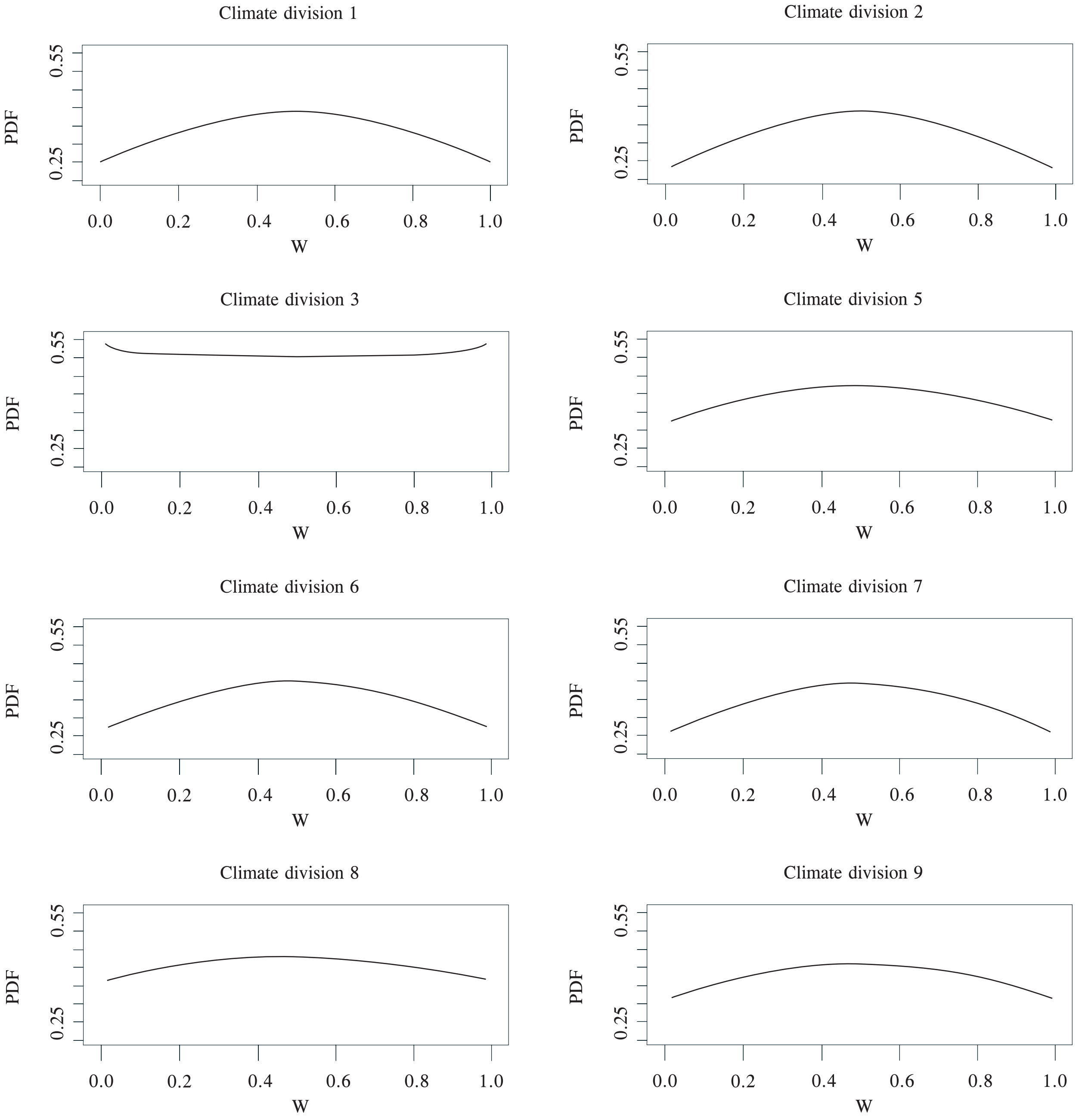

Figure 1. Fitted values of the pdfs (2) and (3) for the eight climate divisions of the State of Nebraska, USA. (X=drought duration and Y=successive non-drought duration.

Figura 1. Valores ajustados de las fdp (2) y (3) para las ocho divisiones climáticas del Estado de Nebraska, EE. UU. (X=duración de la sequía y Y=duración sucesiva de no-sequía.